

# On non-formality of a simply-connected symplectic 8-manifold

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**Abstract.** We show an alternative construction of the first example of a simply-connected compact symplectic non-formal 8-manifold given in [6]. We also give an alternative proof of its non-formality using higher order Massey products.

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## INTRODUCTION

In [1, 2, 10] Babenko–Taimanov and Rudyak–Tralle give examples of non-formal simply-connected compact symplectic manifolds of any even dimension bigger than or equal to 10. Babenko and Taimanov raise the question of the existence of non-formal simply-connected compact symplectic manifolds of dimension 8, which cannot be constructed with their methods. In [6], it is constructed the first example of a simply-connected compact symplectic 8-dimensional manifold which is non-formal, thereby completing the solution to the question of existence of non-formal symplectic manifolds for all allowable dimensions. This example is constructed by starting with a suitable complex 8-dimensional compact nilmanifold  $M$  which has a symplectic form (but is not Kähler). Then one quotients by a suitable action of the finite group  $\mathbb{Z}_3$  acting symplectically and freely except at finitely many fixed points. This gives a symplectic orbifold  $\hat{M} = M/\mathbb{Z}_3$ , which is non-formal and simply-connected thanks to the choice of  $\mathbb{Z}_3$ -action. The last step is a process of symplectic resolution of singularities to get a smooth symplectic manifold. The symplectic resolution of isolated orbifold singularities has been described in detail in [4]. The non-formality of  $\hat{M}$  is checked via a newly defined product in cohomology. This is a product of Massey type, which is called  $a$ -product, and it is discussed at length in [4].

The purpose of the present note is to give a new description of the symplectic orbifold  $\hat{M}$  defined in [6]. The description presented here is in terms of real nilpotent Lie groups. Secondly, we prove the non-formality of  $\hat{M}$  by using higher order Massey products instead of  $a$ -products. It remains thus open the question of the existence of a smooth 8-manifold with non-zero  $a$ -products but trivial (higher order) Massey products.

## A NILMANIFOLD OF DIMENSION 6

Let  $G$  be the simply connected nilpotent Lie group of dimension 6 defined by the structure equations

$$\begin{aligned} d\beta_i &= 0, & i &= 1, 2 \\ d\gamma_i &= 0, & i &= 1, 2 \\ d\eta_1 &= -\beta_1 \wedge \gamma_1 + \beta_2 \wedge \gamma_1 + \beta_1 \wedge \gamma_2 + 2\beta_2 \wedge \gamma_2, \\ d\eta_2 &= 2\beta_1 \wedge \gamma_1 + \beta_2 \wedge \gamma_1 + \beta_1 \wedge \gamma_2 - \beta_2 \wedge \gamma_2, \end{aligned} \tag{1}$$

where  $\{\beta_i, \gamma_i, \eta_i; 1 \leq i \leq 2\}$  is a basis of the left invariant 1-forms on  $G$ . Because the structure constants are rational numbers, Mal'cev theorem [7] implies the existence of a discrete subgroup  $\Gamma$  of  $G$  such that the quotient space  $N = \Gamma \backslash G$  is compact.

Using Nomizu's theorem [9] we can compute the real cohomology of  $N$ . We get

$$\begin{aligned} H^0(N) &= \langle 1 \rangle, \\ H^1(N) &= \langle [\beta_1], [\beta_2], [\gamma_1], [\gamma_2] \rangle, \\ H^2(N) &= \langle [\beta_1 \wedge \beta_2], [\beta_1 \wedge \gamma_1], [\beta_1 \wedge \gamma_2], [\gamma_1 \wedge \gamma_2], [\beta_1 \wedge \eta_2 - \beta_2 \wedge \eta_1], [\gamma_1 \wedge \eta_2 - \gamma_2 \wedge \eta_1], \\ &\quad [\beta_1 \wedge \eta_1 + \beta_1 \wedge \eta_2 + \beta_2 \wedge \eta_2], [\gamma_1 \wedge \eta_1 + \gamma_1 \wedge \eta_2 + \gamma_2 \wedge \eta_2] \rangle, \\ H^3(N) &= \langle [\beta_1 \wedge \beta_2 \wedge \eta_1], [\beta_1 \wedge \beta_2 \wedge \eta_2], [\gamma_1 \wedge \gamma_2 \wedge \eta_1], [\gamma_1 \wedge \gamma_2 \wedge \eta_2], [\beta_1 \wedge \gamma_1 \wedge (\eta_1 + 2\eta_2)], \\ &\quad [\beta_1 \wedge \gamma_1 \wedge \eta_2 - \beta_1 \wedge \gamma_2 \wedge \eta_1], [\beta_1 \wedge \gamma_2 \wedge \eta_1 - \beta_1 \wedge \gamma_2 \wedge \eta_2], [\beta_2 \wedge \gamma_2 \wedge (\eta_2 + 2\eta_1)], \\ &\quad [\beta_2 \wedge \gamma_2 \wedge \eta_1 - \beta_2 \wedge \gamma_1 \wedge \eta_2], [\beta_2 \wedge \gamma_1 \wedge \eta_2 - \beta_2 \wedge \gamma_1 \wedge \eta_1] \rangle, \\ H^4(N) &= \langle [\beta_1 \wedge \beta_2 \wedge \gamma_1 \wedge \eta_1], [\beta_1 \wedge \beta_2 \wedge \gamma_1 \wedge \eta_2], [\beta_1 \wedge \beta_2 \wedge \gamma_1 \wedge \eta_2], [\beta_1 \wedge \gamma_1 \wedge \gamma_2 \wedge \eta_2], \\ &\quad [\beta_2 \wedge \gamma_1 \wedge \gamma_2 \wedge \eta_2], [\gamma_1 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2], [\beta_1 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2 - \beta_2 \wedge \gamma_1 \wedge \eta_1 \wedge \eta_2], \\ &\quad [\beta_1 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2 + \beta_1 \wedge \gamma_1 \wedge \eta_1 \wedge \eta_2 + \beta_2 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2] \rangle, \\ H^5(N) &= \langle [\beta_1 \wedge \beta_2 \wedge \gamma_1 \wedge \eta_1 \wedge \eta_2], [\beta_1 \wedge \beta_2 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2], [\beta_1 \wedge \gamma_1 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2], \\ &\quad [\beta_2 \wedge \gamma_1 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2] \rangle, \\ H^6(N) &= \langle [\beta_1 \wedge \beta_2 \wedge \gamma_1 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2] \rangle. \end{aligned}$$

We can give a more explicit description of the group  $G$ . As a differentiable manifold  $G = \mathbb{R}^6$ . The nilpotent Lie group structure of  $G$  is given by the multiplication law

$$\begin{aligned} m: \quad G \times G &\longrightarrow G \\ ((y'_1, y'_2, z'_1, z'_2, v'_1, v'_2), (y_1, y_2, z_1, z_2, v_1, v_2)) &\mapsto \begin{pmatrix} y_1 + y'_1, y_2 + y'_2, z_1 + z'_1, z_2 + z'_2, \\ v_1 + v'_1 + (y'_1 - y'_2)z_1 - (y'_1 + 2y'_2)z_2, \\ v_2 + v'_2 - (2y'_1 + y'_2)z_1 + (y'_2 - y'_1)z_2 \end{pmatrix}. \end{aligned} \tag{2}$$

We also need a discrete subgroup, which it could be taken to be  $\mathbb{Z}^6 \subset G$ . However, for later convenience, we shall take the subgroup

$$\Gamma = \{(y_1, y_2, z_1, z_2, v_1, v_2) \in \mathbb{Z}^6 \mid v_1 \equiv v_2 \pmod{3}\} \subset G,$$

and define the nilmanifold

$$N = \Gamma \backslash G.$$

In terms of a (global) system of coordinates  $(y_1, y_2, z_1, z_2, v_1, v_2)$  for  $G$ , the 1-forms  $\beta_i$ ,  $\gamma_i$  and  $\eta_i$ ,  $1 \leq i \leq 2$ , are given by

$$\begin{aligned}\beta_i &= dy_i, & 1 \leq i \leq 2, \\ \gamma_i &= dz_i, & 1 \leq i \leq 2, \\ \eta_1 &= dv_1 - y_1 dz_1 + y_2 dz_1 + y_1 dz_2 + 2y_2 dz_2, \\ \eta_2 &= dv_2 + 2y_1 dz_1 + y_2 dz_1 + y_1 dz_2 - y_2 dz_2.\end{aligned}$$

Note that  $N$  is a principal torus bundle

$$T^2 = \mathbb{Z}\langle(1, 1), (3, 0)\rangle \backslash \mathbb{R}^2 \hookrightarrow N \longrightarrow T^4 = \mathbb{Z}^4 \backslash \mathbb{R}^4,$$

with the projection  $(y_1, y_2, z_1, z_2, v_1, v_2) \mapsto (y_1, y_2, z_1, z_2)$ .

The Lie group  $G$  can be also described as follows. Consider the basis  $\{\mu_i, v_i, \theta_i; 1 \leq i \leq 2\}$  of the left invariant 1-forms on  $G$  given by

$$\begin{aligned}\mu_1 &= \beta_1 + \frac{1+\sqrt{3}}{2}\beta_2, & \mu_2 &= \beta_1 + \frac{1-\sqrt{3}}{2}\beta_2, \\ v_1 &= \gamma_1 + \frac{1+\sqrt{3}}{2}\gamma_2, & v_2 &= \gamma_1 + \frac{1-\sqrt{3}}{2}\gamma_2, \\ \theta_1 &= \frac{2}{\sqrt{3}}\eta_1 + \frac{1}{\sqrt{3}}\eta_2, & \theta_2 &= \eta_2.\end{aligned}$$

Hence, the structure equations can be rewritten as

$$\begin{aligned}d\mu_i &= 0, & 1 \leq i \leq 2, \\ dv_i &= 0, & 1 \leq i \leq 2, \\ d\theta_1 &= \mu_1 \wedge v_1 - \mu_2 \wedge v_2, \\ d\theta_2 &= \mu_1 \wedge v_2 + \mu_2 \wedge v_1.\end{aligned} \tag{3}$$

This means that  $G$  is the complex Heisenberg group  $H_{\mathbb{C}}$ , that is, the complex nilpotent Lie group of complex matrices of the form

$$\begin{pmatrix} 1 & u_2 & u_3 \\ 0 & 1 & u_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

In fact, in terms of the natural (complex) coordinate functions  $(u_1, u_2, u_3)$  on  $H_{\mathbb{C}}$ , we have that the complex 1-forms

$$\mu = du_1, \quad v = du_2, \quad \theta = du_3 - u_2 du_1$$

are left invariant and  $d\mu = dv = 0$ ,  $d\theta = \mu \wedge v$ . Now, it is enough to take  $\mu_1 = \Re(\mu)$ ,  $\mu_2 = \Im(\mu)$ ,  $v_1 = \Re(v)$ ,  $v_2 = \Im(v)$ ,  $\theta_1 = \Re(\theta)$ ,  $\theta_2 = \Im(\theta)$  to recover equations (3), where  $\Re(\mu)$  and  $\Im(\mu)$  denote the real and the imaginary parts of  $\mu$ , respectively.

**Lemma 1** *Let  $\Lambda \subset \mathbb{C}$  be the lattice generated by 1 and  $\zeta = e^{2\pi i/3}$ , and consider the discrete subgroup  $\Gamma_H \subset H_{\mathbb{C}}$  formed by the matrices in which  $u_1, u_2, u_3 \in \Lambda$ . Then there is a natural identification of  $N = \Gamma \backslash G$  with the quotient  $\Gamma_H \backslash H_{\mathbb{C}}$ .*

**Proof** We have constructed above an isomorphism of Lie groups  $G \rightarrow H_{\mathbb{C}}$ , whose explicit equations are

$$(y_1, y_2, z_1, z_2, v_1, v_2) \mapsto (u_1, u_2, u_3),$$

where

$$\begin{aligned} u_1 &= \left( y_1 + \frac{1+\sqrt{3}}{2} y_2 \right) + i \left( y_1 + \frac{1-\sqrt{3}}{2} y_2 \right), \\ u_2 &= \left( z_1 + \frac{1+\sqrt{3}}{2} z_2 \right) + i \left( z_1 + \frac{1-\sqrt{3}}{2} z_2 \right), \\ u_3 &= \frac{1}{\sqrt{3}} (2v_1 + v_2 + 3z_1 y_2 + 3z_2 y_1 + 3z_2 y_2) + i (v_2 + 2z_1 y_1 + z_2 y_1 + z_1 y_2 - z_2 y_2). \end{aligned}$$

Note that the formula for  $u_3$  can be deduced from

$$du_3 - u_2 du_1 = \theta = \left( \frac{2}{\sqrt{3}} \eta_1 + \frac{1}{\sqrt{3}} \eta_2 \right) + i \eta_2.$$

Now the group  $\Gamma \subset G$  corresponds under this isomorphism to

$$\left\{ (u_1, u_2, u_3) \mid u_1, u_2 \in \mathbb{Z} \left\langle 1 + i, \frac{1+\sqrt{3}}{2} + \frac{1-\sqrt{3}}{2} i \right\rangle, u_3 \in \mathbb{Z} \langle 2\sqrt{3}, \sqrt{3} + i \rangle \right\}.$$

Using the isomorphism of Lie groups  $H_{\mathbb{C}} \rightarrow H_{\mathbb{C}}$  given by

$$(u_1, u_2, u_3) \mapsto (u'_1, u'_2, u'_3) = \left( \frac{u_1}{1+i}, \frac{u_2}{1+i}, \frac{u_3}{(1+i)^2} \right),$$

we get that  $u'_1, u'_2, u'_3 \in \Lambda = \mathbb{Z} \langle 1, \zeta \rangle$ , which completes the proof.  $\square$

**Remark 2** If we had considered the discrete subgroup  $\mathbb{Z}^6 \subset G$  instead of  $\Gamma \subset G$ , then we would not have obtained the fact  $u'_3 \in \Lambda$  in the proof of Lemma 1. Note that  $N = \Gamma \backslash G \rightarrow \mathbb{Z}^6 \backslash G$  is a 3 : 1 covering.

Under the identification  $N = \Gamma \backslash G \cong \Gamma_H \backslash H_{\mathbb{C}}$ ,  $N$  becomes the principal torus bundle

$$T^2 = \Lambda \backslash \mathbb{C} \hookrightarrow N \longrightarrow T^4 = \Lambda^2 \backslash \mathbb{C}^2,$$

with the projection  $(u_1, u_2, u_3) \mapsto (u_1, u_2)$ .

## A SYMPLECTIC ORBIFOLD OF DIMENSION 8

We define the 8-dimensional compact nilmanifold  $M$  as the product

$$M = T^2 \times N.$$

By Lemma 1 there is an isomorphism between  $M$  and the manifold  $(\Gamma_H \backslash H_{\mathbb{C}}) \times (\Lambda \backslash \mathbb{C})$  studied in [6, Section 2] (we have to send the factor  $T^2$  of  $M$  to the factor  $\Lambda \backslash \mathbb{C}$ ). Clearly,  $M$  is a principal torus bundle

$$T^2 \hookrightarrow M \xrightarrow{\pi} T^6.$$

Let  $(x_1, x_2)$  be the Lie algebra coordinates for  $T^2$ , so that  $(x_1, x_2, y_1, y_2, z_1, z_2, v_1, v_2)$  are coordinates for the Lie algebra  $\mathbb{R}^2 \times G$  of  $M$ . Then  $\pi(x_1, x_2, y_1, y_2, z_1, z_2, v_1, v_2) = (x_1, x_2, y_1, y_2, z_1, z_2)$ . A basis for the left invariant (closed) 1-forms on  $T^2$  is given as  $\{\alpha_1, \alpha_2\}$ , where  $\alpha_1 = dx_1$  and  $\alpha_2 = dx_2$ . Then  $\{\alpha_i, \beta_i, \gamma_i, \eta_i; 1 \leq i \leq 2\}$  constitutes a (global) basis for the left invariant 1-forms on  $M$ . Note that  $\{\alpha_i, \beta_i, \gamma_i; 1 \leq i \leq 2\}$  is a basis for the left invariant closed 1-forms on the base  $T^6$ . (We use the same notation for the differential forms on  $T^6$  and their pullbacks to  $M$ .) Using the computation of the cohomology of  $N$ , we get that the Betti numbers of  $M$  are:  $b_0(M) = b_8(M) = 1$ ,  $b_1(M) = b_7(M) = 6$ ,  $b_2(M) = b_6(M) = 17$ ,  $b_3(M) = b_5(M) = 30$ ,  $b_4(M) = 36$ . In particular,  $\chi(M) = 0$ , as for any nilmanifold.

Consider the action of the finite group  $\mathbb{Z}_3$  on  $\mathbb{R}^2$  given by

$$\rho(x_1, x_2) = (-x_1 - x_2, x_1),$$

for  $(x_1, x_2) \in \mathbb{R}^2$ ,  $\rho$  being the generator of  $\mathbb{Z}_3$ . Clearly  $\rho(\mathbb{Z}^2) = \mathbb{Z}^2$ , and so  $\rho$  defines an action of  $\mathbb{Z}_3$  on the 2-torus  $T^2 = \mathbb{Z}^2 \backslash \mathbb{R}^2$  with 3 fixed points:  $(0, 0)$ ,  $(\frac{1}{3}, \frac{1}{3})$  and  $(\frac{2}{3}, \frac{2}{3})$ . The quotient space  $T^2/\mathbb{Z}_3$  is the orbifold 2-sphere  $S^2$  with 3 points of multiplicity 3. Let  $x_1, x_2$  denote the natural coordinate functions on  $\mathbb{R}^2$ . Then the 1-forms  $dx_1, dx_2$  satisfy  $\rho^*(dx_1) = -dx_1 - dx_2$  and  $\rho^*(dx_2) = dx_1$ , hence  $\rho^*(-dx_1 - dx_2) = dx_2$ . Thus, we can take the 1-forms  $\alpha_1$  and  $\alpha_2$  on  $T^2$  such that

$$\rho^*(\alpha_1) = -\alpha_1 - \alpha_2, \quad \rho^*(\alpha_2) = \alpha_1. \quad (4)$$

Define the following action of  $\mathbb{Z}_3$  on  $M$ , given, at the level of Lie groups, by  $\rho: \mathbb{R}^2 \times \mathbb{R}^6 \longrightarrow \mathbb{R}^2 \times \mathbb{R}^6$ ,

$$\rho(x_1, x_2, y_1, y_2, z_1, z_2, v_1, v_2) = (-x_1 - x_2, x_1, -y_1 - y_2, y_1, -z_1 - z_2, z_1, -v_1 - v_2, v_1).$$

Note that  $m(\rho(p'), \rho(p)) = \rho(m(p', p))$ , for all  $p, p' \in G$ , where  $m$  is the multiplication map (2) for  $G$ . Also  $\Gamma \subset G$  is stable by  $\rho$  since

$$v_1 \equiv v_2 \pmod{3} \implies -v_1 - v_2 \equiv v_1 \pmod{3}.$$

Therefore there is a induced map  $\rho: M \rightarrow M$ , and this covers the action  $\rho: T^6 \rightarrow T^6$  on the 6-torus  $T^6 = T^2 \times T^2 \times T^2$  (defined as the action  $\rho$  on each of the three factors simultaneously). The action of  $\rho$  on the fiber  $T^2 = \mathbb{Z}\langle(1, 1), (3, 0)\rangle$  has also 3 fixed points:  $(0, 0)$ ,  $(1, 0)$  and  $(2, 0)$ . Hence there are  $3^4 = 81$  fixed points on  $M$ .

**Remark 3** Under the isomorphism  $M \cong (\Gamma_H \backslash H_{\mathbb{C}}) \times (\Lambda \backslash \mathbb{C})$ , we have that the action of  $\rho$  becomes  $\rho(u_1, u_2, u_3) = (\tilde{\zeta}u_1, \tilde{\zeta}u_2, \zeta u_3)$ , where  $\zeta = e^{2\pi i/3}$ . Composing the isomorphism of Lemma 1 with the conjugation  $(u_1, u_2, u_3) \mapsto (v_1, v_2, v_3) = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$  (which is an isomorphism of Lie groups  $H_{\mathbb{C}} \rightarrow H_{\mathbb{C}}$  leaving  $\Gamma_H$  invariant), we have that the action of  $\rho$  becomes  $\rho(v_1, v_2, v_3) = (\zeta v_1, \zeta v_2, \zeta^2 v_3)$ . This is the action used in [6].

We take the basis  $\{\alpha_i, \beta_i, \gamma_i, \eta_i; 1 \leq i \leq 2\}$  of the 1-forms on  $M$  considered above. The 1-forms  $dy_i, dz_i, dv_i, 1 \leq i \leq 2$ , on  $G$  satisfy the following conditions similar to (4):  $\rho^*(dy_1) = -dy_1 - dy_2, \rho^*(dy_2) = dy_1, \rho^*(dz_1) = -dz_1 - dz_2, \rho^*(dz_2) = dz_1, \rho^*(dv_1) = -dv_1 - dv_2, \rho^*(dv_2) = dv_1$ . So

$$\begin{aligned} \rho^*(\alpha_1) &= -\alpha_1 - \alpha_2, & \rho^*(\alpha_2) &= \alpha_1, \\ \rho^*(\beta_1) &= -\beta_1 - \beta_2, & \rho^*(\beta_2) &= \beta_1, \\ \rho^*(\gamma_1) &= -\gamma_1 - \gamma_2, & \rho^*(\gamma_2) &= \gamma_1, \\ \rho^*(\eta_1) &= -\eta_1 - \eta_2, & \rho^*(\eta_2) &= \eta_1. \end{aligned} \tag{5}$$

**Remark 4** If we define the 1-forms  $\alpha_3 = -\alpha_1 - \alpha_2, \beta_3 = -\beta_1 - \beta_2, \gamma_3 = -\gamma_1 - \gamma_2$  and  $\eta_3 = -\eta_1 - \eta_2$ , then we have  $\rho^*(\alpha_1) = \alpha_3, \rho^*(\alpha_2) = \alpha_1, \rho^*(\alpha_3) = \alpha_2$ , and analogously for the others.

Define the quotient space

$$\widehat{M} = M/\mathbb{Z}_3,$$

and denote by  $\varphi : M \rightarrow \widehat{M}$  the projection. It is an orbifold, and it admits the structure of a symplectic orbifold (see [4] for a general discussion on symplectic orbifolds).

**Proposition 5** The 2-form  $\omega$  on  $M$  defined by

$$\omega = \alpha_1 \wedge \alpha_2 + \eta_2 \wedge \beta_1 - \eta_1 \wedge \beta_2 + \gamma_1 \wedge \gamma_2$$

is a  $\mathbb{Z}_3$ -invariant symplectic form on  $M$ . Therefore it induces  $\widehat{\omega} \in \Omega_{\text{orb}}^2(\widehat{M})$ , such that  $(\widehat{M}, \widehat{\omega})$  is a symplectic orbifold.

**Proof** Clearly  $\omega^4 \neq 0$ . Using (5) we have that  $\rho^*(\omega) = (-\alpha_1 - \alpha_2) \wedge \alpha_1 + \eta_1 \wedge (-\beta_1 - \beta_2) + (\eta_1 + \eta_2) \wedge \beta_1 + (-\gamma_1 - \gamma_2) \wedge \gamma_1 = \omega$ , so  $\omega$  is  $\mathbb{Z}_3$ -invariant. Finally,

$$d\omega = d\eta_2 \wedge \beta_1 - d\eta_1 \wedge \beta_2 = (\beta_2 \wedge \gamma_1 - \beta_2 \wedge \gamma_2) \wedge \beta_1 - (-\beta_1 \wedge \gamma_1 + \beta_1 \wedge \gamma_2) \wedge \beta_2 = 0.$$

□

It can be seen (cf. proof of Proposition 2.3 in [6]) that  $\widehat{M}$  is simply connected. Moreover, its cohomology can be computed using that

$$H^*(\widehat{M}) = H^*(M)^{\mathbb{Z}_3}.$$

We get

$$\begin{aligned} H^1(\widehat{M}) &= 0, \\ H^2(\widehat{M}) &= \langle [\alpha_1 \wedge \alpha_2], [\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1], [\alpha_1 \wedge \beta_1 + \alpha_1 \wedge \beta_2 + \alpha_2 \wedge \beta_2], \\ &\quad [\alpha_1 \wedge \gamma_2 - \alpha_2 \wedge \gamma_1], [\alpha_1 \wedge \gamma_1 + \alpha_1 \wedge \gamma_2 + \alpha_2 \wedge \gamma_2], [\beta_1 \wedge \beta_2], [\beta_1 \wedge \gamma_2 - \beta_2 \wedge \gamma_1], \\ &\quad [\beta_1 \wedge \gamma_1 + \beta_1 \wedge \gamma_2 + \beta_2 \wedge \gamma_2], [\beta_1 \wedge \eta_2 - \beta_2 \wedge \eta_1], [\beta_1 \wedge \eta_1 + \beta_1 \wedge \eta_2 + \beta_2 \wedge \eta_2], \\ &\quad [\gamma_1 \wedge \gamma_2], [\gamma_1 \wedge \eta_2 - \gamma_2 \wedge \eta_1], [\gamma_1 \wedge \eta_1 + \gamma_1 \wedge \eta_2 + \gamma_2 \wedge \eta_2] \rangle, \\ H^3(\widehat{M}) &= 0. \end{aligned}$$

**Remark 6** The Euler characteristic of  $\widehat{M}$  can be computed via the formula for finite group action quotients: let  $\Pi$  be the cyclic group of order  $n$ , acting on a space  $X$  almost freely. Then

$$\chi(X/\Pi) = \frac{1}{n}\chi(X) + \sum_p \left(1 - \frac{1}{\#\Pi_p}\right),$$

where  $\Pi_p \subset \Pi$  is the isotropy group of  $p \in X$ . In our case  $\chi(\widehat{M}) = \frac{1}{3}\chi(M) + 81(1 - \frac{1}{3}) = 54$ .

Using this remark and the previous calculation, we get that  $b_1(\widehat{M}) = b_7(\widehat{M}) = 0$ ,  $b_2(\widehat{M}) = b_6(\widehat{M}) = 13$ ,  $b_3(\widehat{M}) = b_5(\widehat{M}) = 0$  and  $b_4(\widehat{M}) = 26$ . Note that  $\widehat{M}$  satisfies Poincaré duality since  $H^*(\widehat{M}) = H^*(M)^{\mathbb{Z}_3}$  and  $H^*(M)$  satisfies Poincaré duality.

## NON-FORMALITY OF THE SYMPLECTIC ORBIFOLD

Formality is a property of the rational homotopy type of a space which is of great importance in symplectic geometry. This is due to the fact that compact Kähler manifolds are formal [5] whilst there are compact symplectic manifolds which are non-formal [11, 3, 6]. A general discussion of the property of formality can be found in [11].

The non-formality of a space can be detected by means of Massey products. Let us recall its definition. The simplest type of Massey product is the triple (also known as ordinary) Massey product. Let  $X$  be a smooth manifold and let  $a_i \in H^{p_i}(X)$ ,  $1 \leq i \leq 3$ , be three cohomology classes such that  $a_1 \cup a_2 = 0$  and  $a_2 \cup a_3 = 0$ . The (triple) Massey product of the classes  $a_i$  is defined as the set

$$\langle a_1, a_2, a_3 \rangle = \{[\alpha_1 \wedge \eta + (-1)^{p_1+1} \xi \wedge \alpha_3] \mid a_i = [\alpha_i], \alpha_1 \wedge \alpha_2 = d\xi, \alpha_2 \wedge \alpha_3 = d\eta\}$$

inside  $H^{p_1+p_2+p_3-1}(X)$ . We say that  $\langle a_1, a_2, a_3 \rangle$  is trivial if  $0 \in \langle a_1, a_2, a_3 \rangle$ .

The definition of higher Massey products is as follows (see [8, 11]). The Massey product  $\langle a_1, a_2, \dots, a_t \rangle$ ,  $a_i \in H^{p_i}(X)$ ,  $1 \leq i \leq t$ ,  $t \geq 3$ , is defined if there are differential forms  $\alpha_{i,j}$  on  $X$ , with  $1 \leq i \leq j \leq t$ , except for the case  $(i, j) = (1, t)$ , such that

$$a_i = [\alpha_{i,i}], \quad d\alpha_{i,j} = \sum_{k=i}^{j-1} \bar{\alpha}_{i,k} \wedge \alpha_{k+1,j}, \quad (6)$$

where  $\bar{\alpha} = (-1)^{\deg(\alpha)}\alpha$ . Then the Massey product is

$$\langle a_1, a_2, \dots, a_t \rangle = \left\{ \left[ \sum_{k=1}^{t-1} \bar{\alpha}_{1,k} \wedge \alpha_{k+1,t} \right] \mid \alpha_{i,j} \text{ as in (6)} \right\} \subset H^{p_1+\dots+p_t-(t-2)}(X).$$

We say that the Massey product is trivial if  $0 \in \langle a_1, a_2, \dots, a_t \rangle$ . Note that for  $\langle a_1, a_2, \dots, a_t \rangle$  to be defined it is necessary that  $\langle a_1, \dots, a_{t-1} \rangle$  and  $\langle a_2, \dots, a_t \rangle$  are defined and trivial.

The existence of a non-trivial Massey product is an obstruction to formality, namely, if  $X$  has a non-trivial Massey product then  $X$  is non-formal.

In the case of an orbifold, Massey products are defined analogously but taking the forms to be *orbifold forms* (see [4, Section 2]).

Now we want to prove the non-formality of the orbifold  $\widehat{M}$  constructed in the previous section. By the results of [11],  $M$  is non-formal since it is a nilmanifold which is not a torus. We shall see that this property is inherited by the quotient space  $\widehat{M} = M/\mathbb{Z}_3$ . For this, we study the Massey products on  $\widehat{M}$ .

**Lemma 7**  *$\widehat{M}$  has a non-trivial Massey product if and only if  $M$  has a non-trivial Massey product with all cohomology classes  $a_i \in H^*(M)$  being  $\mathbb{Z}_3$ -invariant cohomology classes.*

**Proof** We shall do the case of triple Massey products, since the general case is similar. Suppose that  $\langle a_1, a_2, a_3 \rangle$ ,  $a_i \in H^{p_i}(\widehat{M})$ ,  $1 \leq i \leq 3$  is a non-trivial Massey product on  $\widehat{M}$ . Let  $a_i = [\alpha_i]$ , where  $\alpha_i \in \Omega_{\text{orb}}^*(\widehat{M})$ . We pull-back the cohomology classes  $\alpha_i$  via  $\varphi^* : \Omega_{\text{orb}}^*(\widehat{M}) \rightarrow \Omega^*(M)$  to get a Massey product  $\langle [\varphi^*\alpha_1], [\varphi^*\alpha_2], [\varphi^*\alpha_3] \rangle$ . Suppose that this is trivial on  $M$ , then  $\varphi^*\alpha_1 \wedge \varphi^*\alpha_2 = d\xi$ ,  $\varphi^*\alpha_2 \wedge \varphi^*\alpha_3 = d\eta$ , with  $\xi, \eta \in \Omega^*(M)$ , and  $\varphi^*\alpha_1 \wedge \eta + (-1)^{p_1+1}\xi \wedge \varphi^*\alpha_3 = df$ . Then  $\tilde{\eta} = (\eta + \rho^*\eta + (\rho^*)^2\eta)/3$ ,  $\tilde{\xi} = (\xi + \rho^*\xi + (\rho^*)^2\xi)/3$  and  $\tilde{f} = (f + \rho^*f + (\rho^*)^2f)/3$  are  $\mathbb{Z}_3$ -invariant and  $\varphi^*\alpha_1 \wedge \tilde{\eta} + (-1)^{p_1+1}\tilde{\xi} \wedge \varphi^*\alpha_3 = d\tilde{f}$ . Writing  $\tilde{\eta} = \varphi^*\hat{\eta}$ ,  $\tilde{\xi} = \varphi^*\hat{\xi}$ ,  $\tilde{f} = \varphi^*\hat{f}$ , for  $\hat{\eta}, \hat{\xi}, \hat{f} \in \Omega_{\text{orb}}^*(\widehat{M})$ , we get  $\alpha_1 \wedge \hat{\eta} + (-1)^{p_1+1}\hat{\xi} \wedge \alpha_3 = d\hat{f}$ , contradicting that  $\langle a_1, a_2, a_3 \rangle$  is non-trivial.

Conversely, suppose that  $\langle a_1, a_2, a_3 \rangle$ ,  $a_i \in H^{p_i}(M)^{\mathbb{Z}_3}$ ,  $1 \leq i \leq 3$ , is a non-trivial Massey product on  $M$ . Then we can represent  $a_i = [\alpha_i]$  by  $\mathbb{Z}_3$ -invariant differential forms  $\alpha_i \in \Omega^{p_i}(M)$ . Let  $\hat{\alpha}_i$  be the induced form on  $\widehat{M}$ . Then  $\langle [\hat{\alpha}_1], [\hat{\alpha}_2], [\hat{\alpha}_3] \rangle$  is a non-trivial Massey product on  $\widehat{M}$ . For if it were trivial then pulling-back by  $\varphi$ , we would get  $0 \in \langle \varphi^*[\hat{\alpha}_1], \varphi^*[\hat{\alpha}_2], \varphi^*[\hat{\alpha}_3] \rangle = \langle a_1, a_2, a_3 \rangle$ .  $\square$

In our case, all the triple and quintuple Massey products on  $\widehat{M}$  are trivial. For instance, for a Massey product of the form  $\langle a_1, a_2, a_3 \rangle$ , all  $a_i$  should have even degree, since  $H^1(\widehat{M}) = H^3(\widehat{M}) = H^5(\widehat{M}) = H^7(\widehat{M}) = 0$ . Therefore the degree of the cohomology classes in  $\langle a_1, a_2, a_3 \rangle$  is odd, hence they are zero.

Since the dimension of  $\widehat{M}$  is 8, there is no room for sextuple Massey products or higher, since the degree of  $\langle a_1, a_2, \dots, a_s \rangle$  is at least  $s+2$ , as  $\deg a_i \geq 2$ . For  $s=6$ , a sextuple Massey product of cohomology classes of degree 2 would live in the top degree cohomology. For computing an element of  $\langle a_1, \dots, a_6 \rangle$ , we have to choose  $\alpha_{i,j}$  in (6). But then adding a closed form  $\phi$  with  $a_1 \cup [\phi] = \lambda[\widehat{M}] \in H^8(\widehat{M})$  to  $\alpha_{2,6}$  we can get another element of  $\langle a_1, \dots, a_6 \rangle$  which is the previous one plus  $\lambda[\widehat{M}]$ . For suitable  $\lambda$  we get  $0 \in \langle a_1, \dots, a_6 \rangle$ .

The only possibility for checking the non-formality of  $\widehat{M}$  via Massey products is to get a non-trivial quadruple Massey product.

From now on, we will denote by the same symbol a  $\mathbb{Z}_3$ -invariant form on  $M$  and that induced on  $\widehat{M}$ . Notice that the 2 forms  $\gamma_1 \wedge \gamma_2$ ,  $\beta_1 \wedge \beta_2$  and  $\alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_1 + \alpha_2 \wedge \gamma_2$  are  $\mathbb{Z}_3$ -invariant forms on  $M$ , hence they descend to the quotient  $\widehat{M} = M/\mathbb{Z}_3$ . We have the following:



**Proposition 8** *The quadruple Massey product*

$$\langle [\gamma_1 \wedge \gamma_2], [\beta_1 \wedge \beta_2], [\beta_1 \wedge \beta_2], [\alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_1 + \alpha_2 \wedge \gamma_2] \rangle$$

is non-trivial on  $\widehat{M}$ . Therefore, the space  $\widehat{M}$  is non-formal.

**Proof** First we see that

$$\begin{aligned} (\gamma_1 \wedge \gamma_2) \wedge (\beta_1 \wedge \beta_2) &= d\xi, \\ (\beta_1 \wedge \beta_2) \wedge (\alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_1 + \alpha_2 \wedge \gamma_2) &= d\varsigma, \end{aligned}$$

where  $\xi$  and  $\varsigma$  are the differential 3-forms on  $\widehat{M}$  given by

$$\begin{aligned} \xi &= -\frac{1}{6}(\gamma_1 \wedge (\beta_1 \wedge \eta_2 + \beta_2 \wedge \eta_2 + \beta_2 \wedge \eta_1) + \gamma_2 \wedge (\beta_1 \wedge \eta_2 + \beta_1 \wedge \eta_1 + \beta_2 \wedge \eta_1)), \\ \varsigma &= \frac{1}{3}(-\alpha_1 \wedge (\eta_2 \wedge \beta_1 + \eta_1 \wedge \beta_1 + \eta_1 \wedge \beta_2) + \alpha_2 \wedge (\eta_2 \wedge \beta_2 - \eta_1 \wedge \beta_1)). \end{aligned}$$

Therefore, the triple Massey products  $\langle [\gamma_1 \wedge \gamma_2], [\beta_1 \wedge \beta_2], [\beta_1 \wedge \beta_2] \rangle$  and  $\langle [\beta_1 \wedge \beta_2], [\beta_1 \wedge \beta_2], [\alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_1 + \alpha_2 \wedge \gamma_2] \rangle$  are defined, and they are trivial because all the (triple) Massey products on  $\widehat{M}$  are trivial. (Notice that the forms  $\xi$  and  $\varsigma$  are  $\mathbb{Z}_3$ -invariant on  $M$  and so descend to  $\widehat{M}$ .) Therefore, the quadruple Massey product  $\langle [\gamma_1 \wedge \gamma_2], [\beta_1 \wedge \beta_2], [\beta_1 \wedge \beta_2], [\alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_1 + \alpha_2 \wedge \gamma_2] \rangle$  is defined on  $\widehat{M}$ . Moreover, it is trivial on  $\widehat{M}$  if and only if there are differential forms  $f_i \in \Omega^3(\widehat{M})$ ,  $1 \leq i \leq 3$ , and  $g_j \in \Omega^4(\widehat{M})$ ,  $1 \leq j \leq 2$ , such that

$$\begin{aligned} (\gamma_1 \wedge \gamma_2) \wedge (\beta_1 \wedge \beta_2) &= d(\xi + f_1), \\ (\beta_1 \wedge \beta_2) \wedge (\beta_1 \wedge \beta_2) &= df_2, \\ (\beta_1 \wedge \beta_2) \wedge (\alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_1 + \alpha_2 \wedge \gamma_2) &= d(\varsigma + f_3), \\ (\gamma_1 \wedge \gamma_2) \wedge f_2 - (\xi + f_1) \wedge (\beta_1 \wedge \beta_2) &= dg_1, \\ (\beta_1 \wedge \beta_2) \wedge (\varsigma + f_3) - f_2 \wedge (\alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_1 + \alpha_2 \wedge \gamma_2) &= dg_2, \end{aligned}$$

and the 6-form given by

$$\Psi = -(\gamma_1 \wedge \gamma_2) \wedge g_2 - g_1 \wedge (\alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_1 + \alpha_2 \wedge \gamma_2) + (\xi + f_1) \wedge (\varsigma + f_3)$$

defines the zero class in  $H^6(\widehat{M})$ . Clearly  $f_1$ ,  $f_2$  and  $f_3$  are closed 3-forms. Since  $H^3(\widehat{M}) = 0$ , we can write  $f_1 = df'_1$ ,  $f_2 = df'_2$  and  $f_3 = df'_3$  for some differential 2-forms  $f'_1$ ,  $f'_2$  and  $f'_3 \in \Omega^2(\widehat{M})$ . Now, multiplying  $[\Psi]$  by the cohomology class  $[\sigma] \in H^2(\widehat{M})$ , where  $\sigma = 2\alpha_1 \wedge \gamma_2 - \alpha_2 \wedge \gamma_1 + \alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_2$  we get

$$\sigma \wedge \Psi = -\frac{1}{3}(\alpha_1 \wedge \alpha_2 \wedge \beta_1 \wedge \beta_2 \wedge \gamma_1 \wedge \gamma_2 \wedge \eta_1 \wedge \eta_2) + d(\sigma \wedge \xi \wedge f'_3 + \sigma \wedge \varsigma \wedge f'_1 + \sigma \wedge f'_1 \wedge df'_3).$$

Hence,  $[2\alpha_1 \wedge \gamma_2 - \alpha_2 \wedge \gamma_1 + \alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_2] \cup [\Psi] \neq 0$ , which implies that  $[\Psi]$  is non-zero in  $H^6(\widehat{M})$ . This proves that the Massey product  $\langle [\gamma_1 \wedge \gamma_2], [\beta_1 \wedge \beta_2], [\beta_1 \wedge \beta_2], [\alpha_1 \wedge \gamma_1 + \alpha_2 \wedge \gamma_1 + \alpha_2 \wedge \gamma_2] \rangle$  is non-trivial, and so  $\widehat{M}$  is non-formal.  $\square$

Finally, there is a way to desingularize  $(\widehat{M}, \widehat{\omega})$  to get a smooth symplectic manifold.

**Theorem 9** *There is a smooth compact symplectic 8-manifold  $(\widetilde{M}, \widetilde{\omega})$  which is simply-connected and non-formal.*

**Proof** By [4, Theorem 3.3], there is a symplectic resolution  $\pi : (\widetilde{M}, \widetilde{\omega}) \rightarrow (\widehat{M}, \widehat{\omega})$ , which consists of a smooth symplectic manifold  $(\widetilde{M}, \widetilde{\omega})$  and a map  $\pi$  which is a diffeomorphism outside the singular points.

To prove the non-formality of  $\widetilde{M}$ , we work as follows. All the forms of the proof of Proposition 8 can be defined on the resolution  $\widetilde{M}$ . Take a  $\mathbb{Z}_3$ -equivariant map  $\psi : M \rightarrow M$  which is the identity outside small balls around the fixed points, and contracts smaller balls onto the fixed points. Substitute the forms  $\vartheta, \tau_i, \kappa, \xi, \dots$  by  $\psi^* \vartheta, \psi^* \tau_i, \psi^* \kappa, \psi^* \xi, \dots$ . Then the corresponding elements in the quadruple Massey product are non-zero, but these forms are zero in a neighbourhood of the fixed points. Therefore they define forms on  $\widetilde{M}$ , by extending them by zero along the exceptional divisors  $E_p = \pi^{-1}(p)$  ( $p \in \widehat{M}$  singular point). Now the proof of Proposition 8 works for  $\widetilde{M}$  with these forms.

Finally, the manifold  $\widetilde{M}$  is simply connected as it is proved in [6, Proposition 2.3] (basically, this follows from the simply-connectivity of  $\widehat{M}$ ).  $\square$

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